The Generalized Liouville's Theorems via Euler-Lagrange Cohomology Groups on Symplectic Manifold

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Abstract

Based on the Euler-Lagrange cohomology groups $H_{EL}^{(2k-1)}(\mathcal{M}^{2n})(1 \leqslant k \leqslant n)$ on symplectic manifold $(\mathcal{M}^{2n}, \omega)$, their properties and a kind of classification of vector fields on the manifold, we generalize Liouville's theorem in classical mechanics to two sequences, the symplectic(-like) and the Hamiltonian-(like) Liouville's theorems. This also generalizes Noether's theorem, since the sequence of symplectic(-like) Liouville's theorems link to the cohomology directly.

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I. INTRODUCTION

It is well known that the theory on symplectic manifolds plays an important role in classical mechanics (see, for example, [1, 2]). And both Lagrangian and Hamiltonian mechanics, which describe mechanical systems with potentials, have been already well established. However, there are many dynamical systems, which have no potentials, such as general volume preserving systems on the symplectic manifolds. How to characterize and describe these systems is an important problem. In addition, the famous Liouville's theorem, which claims that the phase flow of a Hamiltonian system preserves the phase volume, also plays an important role not only in classical mechanics but also in physics. But, what about

those general volume preserving systems without potential on symplectic manifold? This is another important problem.

Recently, the Euler-Lagrange cohomology has first been introduced and discussed in [3, 4] for classical mechanics and field theory in order to explore the relevant topics in the (independent variable(s)) discrete mechanics and field theory including symplectic and multisymplectic algorithms [3, 4, 5]. Based upon these work, we have further found that there is, in fact, a sequence of cohomology groups, called the Euler-Lagrange cohomology groups, on the symplectic manifolds [6], [7], [8]. What has been found in [3, 4, 5] for the classical mechanics with potential is the first one, constructed via the Euler-Lagrange 1-forms. We have also found that these cohomology groups may play some important roles in the classical mechanics with and without potential as well as other dynamical systems on the symplectic manifolds, such as the volume-preserving systems and so on [6, 7, 8].

In this paper, we show that based on these Euler-Lagrange cohomology groups and relevant issues the famous Liouville's theorem in classical mechanics should be generalized to two sequences of theorems: the sequence of symplectic(-like) Liouville's theorems and the Hamiltonian(-like) sequence. For the symplectic(-like) sequence, it does not require the equations of motion of the specified mechanical system hold on the symplectic manifold as the phase space of the system. While for the Hamiltonian(-like) sequence, it does always require on the solutions of the corresponding mechanical system, similar to the famous Liouville's theorem requires on the phase flow of a given Hamiltonian system. In fact, the famous Liouville's theorem is the first one in the Hamiltonian(-like) sequence and corresponds to the image of the first Euler-Lagrange group. Further, the generalization of Liouville's theorem implies a kind of generalization of Noether's theorem.

In order to be self-contained, we first systematically introduce the general definition of these Euler-Lagrange cohomology groups $H_{\mathrm{EL}}^{(2k-1)}(\mathcal{M}^{2n}, \omega)$, with $1 \leqslant k \leqslant n$, on a $2n$-dimensional symplectic manifold $(\mathcal{M}^{2n}, \omega)$ and study their properties in some details. In fact, for each $k \leqslant n$, the Euler-Lagrange cohomology group $H_{\mathrm{EL}}^{(2k-1)}(\mathcal{M}, \omega)$ is a quotient group of the $(2k-1)$st symplectic(-like) vector fields over the $(2k-1)$st Hamiltonian(-like) ones (see the definitions in next section). Thus, these cohomology groups classify the vector fields on symplectic manifold. We show that, for $k = 1$ and $k = n$, they are isomorphic to the corresponding de Rham cohomology groups $H_{\mathrm{dR}}^{(1)}(\mathcal{M}^{2n})$ and $H_{\mathrm{dR}}^{(2n-1)}(\mathcal{M}^{2n})$, respectively. Consequently, due to the Poincaré duality, the first Euler-Lagrange cohomology

group $H_{\mathrm{EL}}^{(1)}(\mathcal{M}^{2n}, \omega)$ and the highest one $H_{\mathrm{EL}}^{(2n-1)}(\mathcal{M}^{2n}, \omega)$ are dual to each other if $\mathcal{M}^{2n}$ is closed. We also show that the other Euler-Lagrange cohomology groups $H_{\mathrm{EL}}^{(2k-1)}(\mathcal{M}^{2n}, \omega)$ for $1 < k < n$ are different from either the de Rham cohomology groups or the harmonic cohomology groups on $(\mathcal{M}^{2n}, \omega)$, in general.

From the cohomological point of view, the ordinary Hamiltonian canonical equations correspond to 1-forms that represent the trivial element in the first Euler-Lagrange group $H_{\mathrm{EL}}^{(1)}(\mathcal{M}^{2n}, \omega)$. Analogous with this fact, it is natural and significant to find the general volume-preserving equations on $(\mathcal{M}^{2n}, \omega)$ from such forms that represent the trivial element in the highest Euler-Lagrange cohomology group $H_{\mathrm{EL}}^{(2n-1)}(\mathcal{M}^{2n}, \omega)$. We have introduced this general kind of volume-preserving equations from this point of view. In general, there are no potentials for these volume-preserving systems described by the equations. Only for the special cases, these equations become the ordinary canonical equations in the Hamilton mechanics. Therefore, the Hamilton mechanics has been generalized to the volume-preserving systems on symplectic manifolds via the cohomology [6, 7, 8].

In classical mechanics, for a Hamiltonian system on $(\mathcal{M}^{2n}, \omega)$, there is a class of phase-area conservation laws for its phase flow g^t. Conservations of both the symplectic 2-form ω and the volume form $\tau = \frac{1}{n!} \omega^n$ on $\mathcal{M}$ are included (see, for example, [1]):

$$\int_{g^t \sigma^2} \omega = \int_{\sigma^2} \omega, \ \forall 2-\mathrm{chain} \ \sigma^2 \subset \mathcal{M}^{2n},$$

$$\vdots$$

$$\int_{g^t \sigma^{2k}} \omega^k = \int_{\sigma^{2k}} \omega^k, \ \forall 2k-\mathrm{chain} \ \sigma^{2k} \subset \mathcal{M}^{2n}, \tag{1}$$

$$\vdots$$

$$\int_{g^t \sigma^{2n}} \omega^n = \int_{\sigma^{2n}} \omega^n, \ \forall 2n-\mathrm{chain} \ \sigma^{2n} \subset \mathcal{M}^{2n},$$

where the power of ω is in the wedge product. The last one is the famous Liouville's theorem. Based on the properties of the cohomological classification of the vector spaces, we generalize the Liouville's theorem. The original Liouville's theorem with respect to the phase flows may be called the Hamiltonian Liouville's theorem. It can be generalized first to the symplectic Liouville's theorem requiring the system to move along a symplectic flow generated by a symplectic vector field. Then we further generalize these two Liouville's theorems to the ones with respect to the $(2k-1)$st Euler-Lagrange group $H_{EL}^{(2k-1)}(\mathcal{M}, \omega)$, $1 < k \leqslant n$, namely

the $(2k-1)$st symplectic(-like) and Hamiltonian(-like) Liouville's theorem, respectively. For the case of $k = n$, the highest Euler-Lagrange cohomology group, it gives the most general symplectic(-like) and Hamiltonian(-like) Liouville's theorems, respectively.

As was just mentioned, this generalization of Liouville's theorem directly leads to a kind of generalization of the famous Noether's theorem for the conservation laws via symmetries. As far as the classical mechanical systems are concerned, all known conservation laws are always associated with certain symmetries and hold on the solution space of the equations of motion of the system. As a matter of fact, corresponding to the kernel and image of the $(2k - 1)$st Euler-Lagrange cohomology group, there is $(2k - 1)$st degree symplectic(-like) and Hamiltonian(-like) area conservation laws, respectively. The former requires the closeness condition of the Euler-Lagrange $(2k-1)$-forms or the vector fields being $(2k-1)$st symplectic(-like). While, the latter requires the exactness of the Euler-Lagrange $(2k - 1)$-forms or the vector fields be $(2k - 1)$st Hamiltonian(-like). In other words, among two kinds of conservation laws for the area preserving of the $2k$-dimensional chain $\sigma^{2k} \subset \mathcal{M}$, the symplectic(-like) ones generalizes Noether's theorem via the Euler-Lagrange cohomology groups.

This paper is arranged as follows. In section 2, we first briefly recall the definition of the first Euler-Lagrange cohomology group on a symplectic manifold $(\mathcal{M}^{2n}, \omega)$ and prove that it is isomorphic to the first de Rham cohomology group on the manifold. Then we introduce the general definition of the $(2k - 1)$st Euler-Lagrange cohomology groups for $1 \leqslant k \leqslant n$ on $(\mathcal{M}^{2n}, \omega)$ and indicate that the highest one is equivalent to the $(2n - 1)$st de Rham cohomology group. We also indicate that in general they are not isomorphic to each other and that they are not isomorphic to either the de Rham cohomology or the harmonic cohomology on $(\mathcal{M}^{2n}, \omega)$. The relative Euler-Lagrange cohomology is also introduced in analog with the relative de Rham cohomology. The general volume-preserving equations are introduced in section 3. Their relations with ordinary canonical equations in the Hamilton mechanics as well as other volume-preserving systems are discussed. It is clear that the general volume-preserving equations are the generalization of the ordinary canonical equations in Hamilton Mechanics. In the section 4, based on these results, we generalize Liouville's theorem in mechanics to the generalized $(2k - 1)$st symplectic(-like) and Hamiltonian(-like) ones, respectively. Finally, we end with some conclusion and remarks.

II. THE EULER-LAGRANGE COHOMOLOGY GROUPS ON SYMPLECTIC MANIFOLD

A. The Hamilton Mechanics, First and Higher Euler-Lagrange Cohomology

As is well known, the ordinary canonical equations can be expressed as

$$-i_{\boldsymbol{X}_H}\omega = \mathrm{d}H, \tag{2}$$

where $\boldsymbol{X}_H$ denotes the Hamiltonian vector field with respect to the Hamiltonian H. Introducing what is called the Euler-Lagrange 1-form

$$E_{\boldsymbol{X}}^{(1)} := -i_{\boldsymbol{X}}\omega, \tag{3}$$

where $\boldsymbol{X}$ is an arbitrary vector field of degree one, the eq. (2) indicates that $E_{\boldsymbol{X}_H}^{(1)}$ is exact. In other words, it belongs to the image part of the (first) Euler-Lagrange cohomology group [5]

$$H_{\mathrm{EL}}^{(1)}(\mathcal{M}, \omega) := Z_{\mathrm{EL}}^{(1)}(\mathcal{M}, \omega)/B_{\mathrm{EL}}^{(1)}(\mathcal{M}, \omega),$$

where

$$Z_{\mathrm{EL}}^{(1)}(\mathcal{M}, \omega) := \{E_{\boldsymbol{X}}^{(1)} \,|\, \mathrm{d}E_{\boldsymbol{X}}^{(1)} = 0\} = \ker(\mathrm{d}) \cap \Omega^1(\mathcal{M}),$$
$$B_{\mathrm{EL}}^{(1)}(\mathcal{M}, \omega) := \{E_{\boldsymbol{X}}^{(1)} \,|\, E_{\boldsymbol{X}}^{(1)} = \mathrm{d}\beta\} = \mathrm{im}(\mathrm{d}) \cap \Omega^1(\mathcal{M}).$$

On the other hand, the vector fields corresponding to the kernel part are (degree one) symplectic by definition (See, e.g., [9]). Therefore, the cohomology can also equivalently be defined as

$$H_{\mathrm{EL}}^{(1)}(\mathcal{M}, \omega) := \mathcal{X}_{\mathrm{S}}^{(1)}(\mathcal{M}, \omega)/\mathcal{X}_{\mathrm{H}}^{(1)}(\mathcal{M}, \omega),$$

due to $Z_{\mathrm{EL}}^{(1)}(\mathcal{M}, \omega) \cong \mathcal{X}_{\mathrm{S}}^{(1)}(\mathcal{M}, \omega)$, $B_{\mathrm{EL}}^{(1)}(\mathcal{M}, \omega) \cong \mathcal{X}_{\mathrm{H}}^{(1)}(\mathcal{M}, \omega)$, where the degree one of the vector fields indicates that they correspond to the Euler-Lagrange 1-forms.

In general, for $1 \leqslant k \leqslant n$, we may define the Euler-Lagrange $(2k-1)$-forms

$$E_{\boldsymbol{X}}^{(2k-1)} := -i_{\boldsymbol{X}}\omega^k, \tag{4}$$

and define the sets of (degree $2k-1$) symplectic(-like) and Hamiltonian(-like) vector fields, respectively, as

$$\mathcal{X}_{\mathrm{S}}^{(2k-1)}(\mathcal{M}, \omega) := \{\, \boldsymbol{X} \in \mathcal{X}(\mathcal{M}) \,|\, \mathrm{d}E_{\boldsymbol{X}}^{(2k-1)} = 0 \,\}, \tag{5}$$
$$\mathcal{X}_{\mathrm{H}}^{(2k-1)}(\mathcal{M}, \omega) := \{\, \boldsymbol{X} \in \mathcal{X}(\mathcal{M}) \,|\, E_{\boldsymbol{X}}^{(2k-1)} \text{ is exact}\,\}.$$

Here $\mathcal{X}(\mathcal{M})$ denotes the space of vector fields on $\mathcal{M}$. It is easy to prove that

$$\mathcal{X}_{\mathrm{S}}^{(1)}(\mathcal{M},\omega) \subseteq \ldots \subseteq \mathcal{X}_{\mathrm{S}}^{(2k-1)}(\mathcal{M},\omega) \subseteq \mathcal{X}_{\mathrm{S}}^{(2k+1)}(\mathcal{M},\omega) \subseteq \ldots \subseteq \mathcal{X}_{\mathrm{S}}^{(2n-1)}(\mathcal{M},\omega),$$

$$\mathcal{X}_{\mathrm{H}}^{(1)}(\mathcal{M},\omega) \subseteq \ldots \subseteq \mathcal{X}_{\mathrm{H}}^{(2k-1)}(\mathcal{M},\omega) \subseteq \mathcal{X}_{\mathrm{H}}^{(2k+1)}(\mathcal{M},\omega) \subseteq \ldots \subseteq \mathcal{X}_{\mathrm{H}}^{(2n-1)}(\mathcal{M},\omega), \quad (6)$$

$$\mathcal{X}_{\mathrm{H}}^{(2k-1)}(\mathcal{M},\omega) \subseteq \mathcal{X}_{\mathrm{S}}^{(2k-1)}(\mathcal{M},\omega).$$

In fact, the (symplectic-like) vector fields $\mathcal{X}_{\mathrm{S}}^{(2k-1)}(\mathcal{M},\omega)$ is a Lie algebra under the commutation bracket of vector fields, and $\mathcal{X}_{\mathrm{H}}^{(2k-1)}(\mathcal{M},\omega)$ is an ideal of $\mathcal{X}_{\mathrm{S}}^{(2k-1)}(\mathcal{M},\omega)$ because

$$[\mathcal{X}_{\mathrm{S}}^{(2k-1)}(\mathcal{M},\omega), \mathcal{X}_{\mathrm{S}}^{(2k-1)}(\mathcal{M},\omega)] \subseteq \mathcal{X}_{\mathrm{H}}^{(2k-1)}(\mathcal{M},\omega). \quad (7)$$

It is clear that, for $k = n$, $\mathcal{X}_{\mathrm{S}}^{(2n-1)}(\mathcal{M},\omega)$ is the Lie algebra of the (symplectic-like) volume-preserving vector fields, including all other Lie algebras just mentioned as its subalgebras.

The quotient Lie algebra

$$H_{\mathrm{EL}}^{(2k-1)}(\mathcal{M},\omega) := \mathcal{X}_{\mathrm{S}}^{(2k-1)}(\mathcal{M},\omega)/\mathcal{X}_{\mathrm{H}}^{(2k-1)}(\mathcal{M},\omega) \quad (8)$$

is called the $(2k-1)st$ *Euler-Lagrange cohomology group*. It is called and treated as a group because it is an Abelian Lie algebra.

On the other hand, for each k $(1 \leqslant k \leqslant n)$, the Euler-Lagrange $(2k-1)$-forms $E_{\mathbf{X}}^{(2k-1)}$ as well as the kernel and image spaces of them with respect to d may be introduced:

$$E_{\mathbf{X}}^{(2k-1)}(\mathcal{M}) := -i_{\mathbf{X}}(\omega^k), \quad \mathbf{X} \in \mathcal{X}(\mathcal{M},\omega); \quad (9)$$

$$Z_{\mathrm{EL}}^{(2k-1)}(\mathcal{M},\omega) := \{E_{\mathbf{X}}^{(2k-1)} | \ \mathrm{d}E_{\mathbf{X}}^{(2k-1)} = 0\}; \quad (10)$$

$$B_{\mathrm{EL}}^{(2k-1)}(\mathcal{M},\omega) := \{E_{\mathbf{X}}^{(2k-1)} | \ E_{\mathbf{X}}^{(2k-1)} \text{ is exact}\}. \quad (11)$$

The $(2k-1)st$ *Euler-Lagrange cohomology group* may also be equivalently defined as

$$H_{\mathrm{EL}}^{(2k-1)}(\mathcal{M},\omega) := Z_{\mathrm{EL}}^{(2k-1)}(\mathcal{M},\omega)/B_{\mathrm{EL}}^{(2k-1)}(\mathcal{M},\omega). \quad (12)$$

The equivalence between (8) and (12) is a corollary of the following lemma:

Lemma 1. *Let $x \in \mathcal{M}$ be an arbitrary point and $\mathbf{X} \in T_x\mathcal{M}$. Then, for each $1 \leqslant k \leqslant n$, $i_{\mathbf{X}}(\omega^k) = 0$ if and only if $\mathbf{X} = 0$.*

B.   The Spaces $\mathcal{X}_{\mathrm{S}}^{(2k-1)}(\mathcal{M}, \omega)$ and Highest Euler-Lagrange Cohomology

In order to investigate the properties of the Euler-Lagrange cohomology groups, it is needed to introduce some operators first.

For a point $x \in \mathcal{M}$, the symplectic form ω can be locally expressed as the well-known formula $\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i$ in the Darboux coordinates (q, p). Then let us introduce a well defined linear map on $\Lambda_x^*(\mathcal{M})$:

$$\hat{f} := i_{\frac{\partial}{\partial q^i}} i_{\frac{\partial}{\partial p_i}}. \tag{13}$$

Note that $\hat{f} = 0$ when acting on $\Lambda^1(T_x^*\mathcal{M})$ or $\Lambda^0(T_x^*\mathcal{M})$. And a map $\hat{f}$ can be defined on the exterior bundle $\Lambda^*(\mathcal{M})$. Further, a linear homomorphism, denoted also by $\hat{f}$, can be obtained on $\Omega^*(\mathcal{M})$, the space of differential forms. Especially, we have the identity

$$\hat{f}\omega = n. \tag{14}$$

Another two operators

$$\hat{e} : \Lambda_x^*(\mathcal{M}) \longrightarrow \Lambda_x^*(\mathcal{M}), \quad \alpha \longmapsto \hat{e}\alpha = \alpha \wedge \omega \tag{15}$$

and

$$\hat{h} : \Lambda^k(T_x^*\mathcal{M}) \longrightarrow \Lambda^k(T_x^*\mathcal{M}), \quad \alpha \longmapsto \hat{h}\alpha = (k - n)\,\alpha \tag{16}$$

can also be defined at each $x \in \mathcal{M}$.

Lemma 2. *The operators $\hat{e}$, $\hat{f}$ and $\hat{h}$ on $\Lambda_x^*(\mathcal{M})$ satisfy*

$$[\hat{h}, \hat{e}] = 2\,\hat{e}, \quad [\hat{h}, \hat{f}] = -2\,\hat{f}, \quad [\hat{e}, \hat{f}] = \hat{h}, \quad \forall\, x \in \mathcal{M}. \tag{17}$$

Proof. These relations can be verified directly. Here is a trickier proof.

First we define some "fermionic" operators on $\Lambda_x^*(\mathcal{M})$

$$\begin{aligned} \psi_i := i_{\frac{\partial}{\partial q^i}}, \qquad & \psi^i := i_{\frac{\partial}{\partial p_i}}, \\ \chi_i : \alpha \longmapsto \mathrm{d}p_i \wedge \alpha, \qquad & \chi^i : \alpha \longmapsto \mathrm{d}q^i \wedge \alpha. \end{aligned} \tag{18}$$

For these operators, it is easy to verify that the non-vanishing anti-commutators are

$$\{\psi_i, \chi^j\} = \delta_i^j, \qquad \{\psi^i, \chi_j\} = \delta_j^i. \tag{19}$$

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Given an integer $0 \leqslant k \leqslant 2n$, we can check that, for any $\alpha \in \Lambda^k(T_x^*\mathcal{M})$, $(\chi_i \psi^i + \chi^i \psi_i)\alpha = k\,\alpha$. Therefore,

$$\hat{h} = \chi_i \psi^i + \chi^i \psi_i - n. \tag{20}$$

According to the definitions,

$$\hat{e} = \chi_i \chi^i, \qquad \hat{f} = \psi_i \psi^i. \tag{21}$$

Then the relations in eqs. (17) can be obtained when $\hat{e}$, $\hat{f}$ and $\hat{h}$ are viewed as bosonic operators. $\qquad\square$

For a point $x \in \mathcal{M}$ the following formulas can be derived recursively:

$$[\hat{e}^k, \hat{f}] = k\,\hat{e}^{k-1}(\hat{h} + k - 1), \qquad [\hat{e}, \hat{f}^k] = k\,\hat{f}^{k-1}(\hat{h} - k + 1), \tag{22}$$

where k is an arbitrary positive integer. Then there is the lemma:

Lemma 3. *Let α be a 2-form. If $\hat{e}^k \alpha = 0$ for some $k < n - 1$, then $\alpha = 0$.*

Proof. Applying both sides of the first equation in (22) on α, we have

$$\hat{e}^k \hat{f}\alpha = k\,(k - n + 1)\,\hat{e}^{k-1}\alpha. \tag{23}$$

Since $\hat{f}\alpha$ is a number at each point, the left hand side is $(\hat{f}\alpha)\,\omega^k$. Applying $\hat{e}$ on both sides, we get

$$(\hat{f}\alpha)\,\omega^{k+1} = k(k - n + 1)\,\hat{e}^k\alpha = 0.$$

Since $k + 1 < n$, we have $\hat{f}\alpha = 0$. Now the identity (23) becomes

$$k\,(k - n + 1)\,\hat{e}^{k-1}\alpha = 0.$$

Since $k < n - 1$, we obtain $\hat{e}^{k-1}\alpha = 0$. Therefore, the value of k can be reduced by 1, and further it can be eventually reduced to 0. $\qquad\square$

The above lemma implies that the map sending $\alpha \in \Lambda^2(T_x^*\mathcal{M})$ to $\alpha \wedge \omega^{n-2} \in \Lambda^{2n-2}(T_x^*\mathcal{M})$ is an isomorphism.

We have indicated in previous subsection that $\mathcal{X}_{\mathrm{S}}(\mathcal{M}, \omega) = \mathcal{X}_{\mathrm{S}}^{(1)}(\mathcal{M}, \omega) \subseteq \mathcal{X}_{\mathrm{S}}^{(2k-1)}(\mathcal{M}, \omega)$ for each possible k. The following theorem tells us much more.

Theorem 1. *Let $(\mathcal{M}, \omega)$ be a 2n-dimensional symplectic manifold with $n \geqslant 2$. Then, for each $k \in \{1, 2, \ldots, n-1\}$,*

$$\mathcal{X}_{\mathrm{S}}^{(2k-1)}(\mathcal{M}, \omega) = \mathcal{X}_{\mathrm{S}}(\mathcal{M}, \omega). \tag{24}$$

Proof. We need only to prove that $\mathcal{X}_{\mathrm{S}}^{(2k-1)}(\mathcal{M}, \omega) \subseteq \mathcal{X}_{\mathrm{S}}(\mathcal{M}, \omega)$ for each $k \in \{1, 2, \ldots, n-1\}$.

In fact, for any $\boldsymbol{X} \in \mathcal{X}_{\mathrm{S}}^{(2k-1)}(\mathcal{M}, \omega)$,

$$\mathcal{L}_{\boldsymbol{X}}(\omega^k) = k\,(\mathcal{L}_{\boldsymbol{X}}\omega) \wedge \omega^{k-1} = 0.$$

Since $0 \leqslant k-1 \leqslant n-2$ while $\mathcal{L}_{\boldsymbol{X}}\omega$ is a 2-form, we can use Lemma 3 pointwisely. This yields $\mathcal{L}_{\boldsymbol{X}}\omega = 0$. Thus, $\boldsymbol{X} \in \mathcal{X}_{\mathrm{S}}(\mathcal{M}, \omega)$. This proves $\mathcal{X}_{\mathrm{S}}^{(2k-1)}(\mathcal{M}, \omega) \subseteq \mathcal{X}_{\mathrm{S}}(\mathcal{M}, \omega)$ when $1 \leqslant k \leqslant n-1$. $\qquad\square$

Corollary 1.1. *For each $k < n$, $[\mathcal{X}_{\mathrm{S}}^{(2k-1)}(\mathcal{M}, \omega), \mathcal{X}_{\mathrm{S}}^{(2k-1)}(\mathcal{M}, \omega)] \subseteq \mathcal{X}_{\mathrm{H}}(\mathcal{M}, \omega)$.*

As was implied by Lemma 1, the map $\mathcal{X}(\mathcal{M}) \longrightarrow \Omega^{2n-1}(\mathcal{M}), \boldsymbol{X} \longmapsto i_{\boldsymbol{X}}(\omega^n)$ is a linear isomorphism. From $\mathcal{L}_{\boldsymbol{X}}(\omega^n) = \mathrm{d}i_{\boldsymbol{X}}(\omega^n)$, it follows that $\mathcal{X}_{\mathrm{S}}^{(2n-1)}(\mathcal{M}, \omega)$ is isomorphic to $Z^{(2n-1)}(\mathcal{M})$, the space of closed $(2n-1)$-forms. Lemma 1 also implies that $\mathcal{X}_{\mathrm{H}}^{(2n-1)}(\mathcal{M}, \omega)$ is isomorphic to $B^{(2n-1)}(\mathcal{M})$, the space of exact $(2n-1)$-forms. These can be summarized as in the following theorem:

Theorem 2. *The linear map $\mu_n : \mathcal{X}(\mathcal{M}) \longrightarrow \Omega^{2n-1}(\mathcal{M}), \boldsymbol{X} \longmapsto i_{\boldsymbol{X}}(\omega^n)$ is an isomorphism. Under this isomorphism, $\mathcal{X}_{\mathrm{S}}^{(2n-1)}(\mathcal{M}, \omega)$ and $\mathcal{X}_{\mathrm{H}}^{(2n-1)}(\mathcal{M}, \omega)$ are isomorphic to $Z^{(2n-1)}(\mathcal{M})$ and $B^{(2n-1)}(\mathcal{M})$, respectively.*

Corollary 2.1. *The $(2n-1)$st Euler-Lagrange cohomology group $H_{\mathrm{EL}}^{(2n-1)}(\mathcal{M}, \omega)$ is linearly isomorphic to $H_{\mathrm{dR}}^{(2n-1)}(\mathcal{M})$, the $(2n-1)$st de Rham cohomology group.*

When $\mathcal{M}$ is closed, $H_{\mathrm{EL}}^{(2n-1)}(\mathcal{M}, \omega)$ is linearly isomorphic to the dual space of $H_{\mathrm{EL}}^{(1)}(\mathcal{M}, \omega)$, because $H_{\mathrm{dR}}^{(k)}(\mathcal{M}) \cong (H_{\mathrm{dR}}^{(2n-k)}(\mathcal{M}))^*$ for such kind of manifolds. If $\mathcal{M}$ is not compact, this relation cannot be assured.

C. The Other Euler-Lagrange Cohomology Groups

Although the first and the last Euler-Lagrange cohomology groups can be identified with the corresponding de Rham cohomology groups, respectively, it is still valuable to know

whether the other Euler-Lagrange cohomology groups are nontrivial and different from corresponding de Rham cohomology groups in general.

In this subsection we will enumerate some examples and properties relative to this problem. We shall point out that, for the torus T^{2n} with the standard symplectic structure ω and $n \geqslant 3$, $H_{\mathrm{EL}}^{(2k-1)}(T^{2n}, \omega)$ is not isomorphic to $H_{\mathrm{dR}}^{(2k-1)}(T^{2n})$ whenever $1 < k < n$ (see, Corollary 3.1). In addition, we shall prove that there is a 6-dimensional symplectic manifold $(\mathcal{M}^6, \omega)$ for which the Euler-Lagrange cohomology group $H_{\mathrm{EL}}^{(3)}(\mathcal{M}^6, \omega)$ is not isomorphic to $H_{\mathrm{EL}}^{(1)}(\mathcal{M}, \omega)$ (see, Theorem 4). Therefore, these indicate that the Euler-Lagrange cohomology groups other than the first and the last ones are some new features of certain given symplectic manifolds.

Let L_α be the homomorphism defined by the cup product with a cohomology class $[\alpha]$, where α is a representative. From the definition, there is an injective homomorphism of vector spaces,

$$\pi_{2k-1} : H_{\mathrm{EL}}^{(2k-1)}(\mathcal{M}, \omega) \longrightarrow H_{\mathrm{dR}}^{(2k-1)}(\mathcal{M}),$$

for each $k \in \{1, 2, \ldots, n\}$ such that the following diagram is commutative:

$$
\begin{array}{ccccccc}
H_{\mathrm{EL}}^{(1)}(\mathcal{M}, \omega) & \longrightarrow & H_{\mathrm{EL}}^{(3)}(\mathcal{M}, \omega) & \longrightarrow & \cdots & \longrightarrow & H_{\mathrm{EL}}^{(2n-3)}(\mathcal{M}, \omega) \\
{\scriptstyle \pi_1} \downarrow & & {\scriptstyle \pi_3} \downarrow & & {\scriptstyle \cdots} \downarrow & & {\scriptstyle \pi_{2n-3}} \downarrow \\
H_{\mathrm{dR}}^{(1)}(\mathcal{M}) & \xrightarrow{L_\omega} & H_{\mathrm{dR}}^{(3)}(\mathcal{M}) & \xrightarrow{L_\omega} \cdots \xrightarrow{L_\omega} & & & H_{\mathrm{dR}}^{(2n-3)}(\mathcal{M}).
\end{array}
\tag{25}
$$

In fact, for an equivalence class $[\boldsymbol{X}]_{(2k-1)} \in H_{\mathrm{EL}}^{(2k-1)}(\mathcal{M}, \omega)$ $(1 \leqslant k \leqslant n)$ with $\boldsymbol{X} \in \mathcal{X}_{\mathrm{S}}^{(2k-1)}(\mathcal{M}, \omega)$ an arbitrary representative, $-\frac{1}{k} i_{\boldsymbol{X}}(\omega^k)$ is a closed $(2k-1)$-form. Thus the de Rham cohomology class of this form can be defined to be $\pi_{2k-1}([\boldsymbol{X}]_{(2k-1)})$. It is easy to verify that this definition is well defined: $\pi_{2k-1}([\boldsymbol{X}]_{(2k-1)})$ does not depend on the choice of the representative $\boldsymbol{X}$ in $[\boldsymbol{X}]_{(2k-1)}$. As for the horizontal maps in the first row of the above diagram, they are induced by the identity map on $\mathcal{X}_{\mathrm{S}}^{(2k-1)}(M, \omega) = \mathcal{X}_{\mathrm{S}}^1(\mathcal{M}, \omega)$ where $k \in \{1, 2, \ldots, n-1\}$. For example, if $[\boldsymbol{X}]_{(2k-1)}$ is an equivalence class in $H_{\mathrm{EL}}^{(2k-1)}(\mathcal{M}, \omega)$ $(k = 1, 2, \ldots, n-1, \ n > 1)$ where $\boldsymbol{X} \in \mathcal{X}_{\mathrm{S}}^{(2k-1)}(\mathcal{M}, \omega)$ is an arbitrary representative, then $[\boldsymbol{X}]_{(2k-1)}$ is mapped to be an equivalence class $[\boldsymbol{X}]_{2k+1}$ in $H_{\mathrm{EL}}^{(2k+1)}(\mathcal{M}, \omega)$. It is also easy to check that this is a well defined homomorphism.

Since π_1 is an isomorphism and the horizontal homomorphisms in the first row are all onto, it follows that

Theorem 3. *For $2 \leqslant k \leqslant n-1$, π_{2k-1} is onto if and only if $L_\omega^{k-1} = L_{\omega^{k-1}}$ from $H_{\mathrm{dR}}^{(1)}(\mathcal{M})$ to $H_{\mathrm{dR}}^{(2k-1)}(\mathcal{M})$ is onto, and $L_{\omega^{k-1}} : H_{\mathrm{dR}}^{(1)}(\mathcal{M}) \longrightarrow H_{\mathrm{dR}}^{(2k-1)}(\mathcal{M})$ is injective if and only if the homomorphism from $H_{\mathrm{EL}}^{1}(\mathcal{M}, \omega)$ to $H_{\mathrm{EL}}^{(2k-1)}(\mathcal{M}, \omega)$ is injective.*

Corollary 3.1. *For $n \geqslant 3$, let $\mathcal{M}$ be the torus T^{2n} with the standard symplectic structure ω. Then, for $1 < k < n$, $H_{\mathrm{EL}}^{(2k-1)}(\mathcal{M}, \omega) \neq H_{\mathrm{dR}}^{(2k-1)}(\mathcal{M})$.*

Proof. As is well known, the de Rham cohomology groups of T^{2n} satisfy

$$\dim H_{\mathrm{dR}}^{(k)}(T^{2n}) = \binom{2n}{k}$$

for each $0 \leqslant k \leqslant 2n$. Therefore, we have $\dim H_{\mathrm{dR}}^{(2k-1)}(T^{2n}) > 2n$ for each $1 < k < n$. On the other hand, due to the fact that the maps in the first row of the diagram (25) are surjective, we have $\dim H_{\mathrm{EL}}^{(2k-1)}(T^{2n}, \omega) \leqslant 2n$ for each $1 < k < n$. So, $\dim H_{\mathrm{dR}}^{(2k-1)}(T^{2n}) > \dim H_{\mathrm{EL}}^{(2k-1)}(T^{2n}, \omega)$. $\qquad\square$

Further, we will show that there are some symplectic manifolds for which $H_{\mathrm{EL}}^{(2k-1)} \neq H_{\mathrm{EL}}^{(1)}$.

Recall that on an n-dimensional Lie group G there exists a basis that consists of n left-invariant vector fields $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$. They form the Lie algebra $\mathfrak{g}$ of G. Let $[\boldsymbol{X}_i, \boldsymbol{X}_j] = -c_{ij}^k \boldsymbol{X}_k$ with the structural constants c_{ij}^k of $\mathfrak{g}$. Let $\{\theta^k\}$ be the left-invariant dual basis. Then they satisfy the equation

$$\mathrm{d}\theta^k = \frac{1}{2} c_{ij}^k \theta^i \wedge \theta^j, \quad k = 1, \ldots, n. \tag{26}$$

G is called a *nilpotent* Lie group if $\mathfrak{g}$ is nilpotent. A *nilmanifold* is defined to be a closed manifold M of the form G/Γ where G is a simply connected nilpotent group and Γ is a discrete subgroup of G. It is well known that Γ determines G and is determined by G uniquely up to isomorphisms (provided that Γ exists) [10, 11].

There are three important facts for the compact nilmanifolds [12]:

1. Let $\mathfrak{g}$ be a nilpotent Lie algebra with structural constants c_{ij}^k with respect to some basis, and let $\{\theta^1, \ldots, \theta^n\}$ be the dual basis of $\mathfrak{g}^*$. Then in the Chevalley-Eilenberg complex $(\Lambda^* \mathfrak{g}^*, \mathrm{d})$ we have

$$\mathrm{d}\theta^k = \sum_{1 \leqslant i < j < k} c_{ij}^k \theta^i \wedge \theta^j, \quad k = 1, \ldots, n. \tag{27}$$

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2. Let $\mathfrak{g}$ be the Lie algebra of a simply connected nilpotent Lie group G. Then, by Malcev's theorem [10], G admits a lattice if and only if $\mathfrak{g}$ admits a basis such that all the structural constants are rational.

3. By Nomizu's theorem, the Chevalley-Eilenberg complex $(\Lambda^* \mathfrak{g}^*, d)$ of $\mathfrak{g}$ is quasi-isomorphic to the de Rham complex of G/Γ. In particular,

$$H^*(G/\Gamma) \cong H^*(\Lambda^* \mathfrak{g}^*, d) \tag{28}$$

and any cohomology class $[\alpha] \in H^k(G/\Gamma)$ contains a homogeneous representative α. Here we call the form α homogeneous if the pullback of α to G is left-invariant.

These results allow us to compute cohomology invariants of nilmanifolds in terms of the Lie algebra $\mathfrak{g}$, and this simplifies the calculations.

Theorem 4. *There exists a 6-dimensional symplectic nilmanifold $(\mathcal{M}^6, \omega)$ such that*

$$H_{\mathrm{EL}}^{(3)}(\mathcal{M}, \omega) \neq H_{\mathrm{EL}}^{(1)}(\mathcal{M}, \omega).$$

Proof. To define the manifold $\mathcal{M}$, it suffices to give the Lie algebra. $\mathfrak{g}$ is a 6-dimensional Lie algebra generated by the generators $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_6$ with Lie bracket given by

$$[\boldsymbol{X}_i, \boldsymbol{X}_j] = - \sum_{1 \leqslant i < j < k} c_{ij}^k \boldsymbol{X}_k.$$

This Lie algebra gives a unique nilmanifold $\mathcal{M}$ by the above information on nilmanifolds. The Chevalley-Eilenberg complex $(\Lambda^* \mathfrak{g}^*, d)$ of $\mathfrak{g}$, which is used to calculate the de Rham cohomology of $\mathcal{M}$, is as in the following.

Let $A = \Lambda^*(\theta^1, \ldots, \theta^6)$ be generated by the 1-forms θ^i, $1 \leqslant i \leqslant 6$. Their differentials are given by the following formulas:

$$\begin{aligned}
d\theta^1 &= 0, & d\theta^2 &= 0, \\
d\theta^3 &= 0, & d\theta^4 &= \theta^1 \wedge \theta^2, \\
d\theta^5 &= \theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3, & d\theta^6 &= \theta^1 \wedge \theta^5 + \theta^3 \wedge \theta^4.
\end{aligned}$$

Furthermore, the symplectic form ω on $\mathcal{M}$ is induced by $F = \theta^1 \wedge \theta^6 + \theta^2 \wedge \theta^4 + \theta^3 \wedge \theta^5 \in A$. ω is a symplectic form since $F \wedge F \wedge F$ is nontrivial by an easy calculation.

It is not difficult to show

$$H_{\mathrm{dR}}^{(1)}(\mathcal{M}) = \mathbb{R}^3 = \mathrm{span}\{[\theta^1], [\theta^2], [\theta^3]\}$$

and

$$H_{\mathrm{dR}}^{(2)}(\mathcal{M}) = \mathbb{R}^4 = \mathrm{span}\{[\theta^1 \wedge \theta^3], [\theta^1 \wedge \theta^4], [\theta^2 \wedge \theta^4], [F]\}.$$

To prove the theorem, it is sufficient to prove that $\omega \wedge \theta^1$ is cohomologically trivial, according to Theorem 3. This follows the equation

$$F \wedge \theta^1 = \theta^1 \wedge \theta^2 \wedge \theta^4 + \theta^1 \wedge \theta^3 \wedge \theta^5 = \mathrm{d}(\theta^2 \wedge \theta^5 + \theta^3 \wedge \theta^6),$$

which can be checked easily and thus completes the proof. $\qquad\square$

D. The Euler-Lagrange Cohomology and the Harmonic Cohomology

On a given symplectic manifold $(\mathcal{M}, \omega)$, there also exists the harmonic cohomology in addition to the de Rham cohomology. In this subsection we explore the relation between the Euler-Lagrange cohomology and the harmonic cohomology on $(\mathcal{M}, \omega)$, and show that they are different from each other in general.

Given a smooth symplectic manifold $(\mathcal{M}, \omega)$, let the $*$-operator

$$* : \Omega^k(\mathcal{M}) \to \Omega^{2n-k}(\mathcal{M})$$

be introduced [13] in analog with the $*$-operator on a Riemannian manifold. Define

$$\delta : \Omega^k(\mathcal{M}) \to \Omega^{k-1}(\mathcal{M}), \quad \delta(\alpha) = (-1)^{k+1} * \mathrm{d} * \alpha.$$

It turns out to be that $\delta = [i(\Pi), \mathrm{d}]$ [13, 14], where $i(\Pi)$ is, in fact, the operator $\hat{f}$ introduced in subsection II B.

Remark 1: The operator $\delta = - * \mathrm{d}*$ was also considered by Libermann (see [15]). Koszul [16] introduced the operator $\delta = [\mathrm{d}, i(\Pi)]$ for Poisson manifolds. Brylinski [14] proved that these operators coincide with each other.

Definition: A form α on a symplectic manifold $(\mathcal{M}, \omega)$ is called *symplectically harmonic* if $d\alpha = 0 = \delta\alpha$.

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We denote by $\Omega_{\mathrm{hr}}^k(M)$ the linear space of symplectically harmonic k-forms. Unlike the Hodge theory, there are non-zero exact symplectically harmonic forms. Now, following Brylinski [14], we define symplectically harmonic cohomology $H_{\mathrm{hr}}^*(\mathcal{M}, \omega)$ by setting

$$H_{\mathrm{hr}}^k(\mathcal{M}, \omega) := \Omega_{\mathrm{hr}}^k(\mathcal{M})/(\mathrm{im}(\mathrm{d}) \cap \Omega_{\mathrm{hr}}^k(\mathcal{M})).$$

Therefore, $H_{\mathrm{hr}}^k(\mathcal{M}, \omega) \subset H_{\mathrm{dR}}^k(\mathcal{M})$.

We would like to know if the symplectically harmonic cohomology and the Euler-Lagrange cohomology are isomorphic to each other. The following result answers no to this question.

Theorem 5. *Let $\mathcal{M}$ be the $2n$-dimensional torus T^{2n} with standard symplectic structure. Then $H_{\mathrm{EL}}^{(2k-1)}(\mathcal{M}, \omega)$ and $H_{\mathrm{hr}}^{(2k-1)}(\mathcal{M}, \omega)$ are not the same for $1 < k < n$.*

This is because in this case the symplectically harmonic cohomology are the same as the de Rham cohomology and now the result follows from Corollary 3.1.

E. The Relative Euler-Lagrange Cohomology

Let us now propose a definition of relative Euler-Lagrange cohomology. That is the combination of the above definition of Euler-Lagrange cohomology and the usual definition of relative de Rham cohomology.

Let $\mathcal{M}$ be a symplectic $2n$-manifold and $i : \mathcal{N} \longrightarrow \mathcal{M}$ be an embedded submanifold. Recall that the usual relative de Rham forms are defined by

$$\Omega^k(i) = \Omega^k(\mathcal{M}) \oplus \Omega^{k-1}(\mathcal{N})$$

where $\Omega^{k-1}(\mathcal{N})$ is the group of $(k-1)$-forms on $\mathcal{N}$. The differential on $\Omega^*(i)$ is given by

$$\mathrm{d}(\theta_1, \theta_2) = (\mathrm{d}\theta_1, i^*\theta_1 - \mathrm{d}\theta_2). \tag{29}$$

Definition: Define

$$\Omega_{\mathrm{EL}}^{(2k-1)}(i, \omega) = \Omega_{\mathrm{EL}}^{(2k-1)}(\mathcal{M}, \omega) \oplus \Omega^{2k-2}(\mathcal{N})$$

where $\Omega_{\mathrm{EL}}^{(2k-1)}(\mathcal{M}, \omega) = \{i_{\boldsymbol{X}}(\omega^k) \,|\, \boldsymbol{X} \in \mathcal{X}(\mathcal{M})\}$. The *relative Euler-Lagrange cohomology* will be defined as

$$H_{\mathrm{EL}}^{(2k-1)}(i, \omega) = \frac{\{(\theta_1, \theta_2) \in \Omega_{\mathrm{EL}}^k(i, \omega) \,|\, \mathrm{d}(\theta_1, \theta_2) = 0\}}{\{(\theta_1, \theta_2) \,|\, (\theta_1, \theta_2) = \mathrm{d}(\theta_1', \theta_2')\}}.$$

Let us consider an example for the relative Euler-Lagrange cohomology. Let $\mathcal{M} = \mathbb{R}^{2n}$, $\mathcal{N} = T^n$ and $i : T^n \longrightarrow \mathbb{R}^{2n}$ be the inclusion.

Proposition 1. $H_{\mathrm{EL}}^{(2k-1)}(i) = H_{\mathrm{dR}}^{(2k-2)}(T^n)$.

Proof. There is the obvious linear map $\Omega^{2k-2}(T^n) \longrightarrow \Omega_{\mathrm{EL}}^{2k-1}(i)$ given by $\theta \longmapsto (0, \theta)$. When $\theta = \mathrm{d}\alpha \in \Omega^{2k-2}(T^{2n})$ is exact, $(0, \theta) = \mathrm{d}(0, -\alpha) \in \Omega_{\mathrm{EL}}^{2k-1}(i)$ is also exact. Therefore a linear map $f : H_{\mathrm{dR}}^{(2k-2)}(T^{2n}) \longrightarrow H_{\mathrm{EL}}^{(2k-1)}(i)$, $[\theta] \longmapsto [(0, \theta)]$ can be induced.

This map f is an injection. In fact, if $(0, \theta) = \mathrm{d}(\alpha_1, \alpha_2)$, then $\mathrm{d}\alpha_1 = 0$, $\theta = i^*(\alpha_1) - \mathrm{d}\alpha_2$. Thus θ is exact by the fact that any closed form α_1 on $\mathbb{R}^{2n}$ is exact.

The map f is also an epimorphism: For any closed (θ_1, θ_2), it is in the same cohomology class of the element $(0, \theta_2 - i^*(\alpha_1) + \mathrm{d}\alpha_2) = (\theta_1, \theta_2) - \mathrm{d}(\alpha_1, \alpha_2)$, where $\theta_1 = \mathrm{d}\alpha_1$ and α_2 is any form on T^n. Obviously, $\theta_2 - i^*(\alpha_1) + \mathrm{d}\alpha_2$ is closed and $f([\theta_2 - i^*(\alpha_1) + \mathrm{d}\alpha_2]) = [(\theta_1, \theta_2)]$. $\square$

Remark 2: Although it is not verified yet, the following statement, if true, will not be a surprise : There exists a symplectic manifold $\mathcal{M}$ and its submanifold $i : \mathcal{N} \longrightarrow \mathcal{M}$ for which the (relative) Euler-Lagrange cohomology is not the corresponding (relative) de Rham cohomology.

Remark 3: For the definition of $H_{\mathrm{EL}}^{(2k-1)}(i, \omega)$, it is also possible to require that (θ_1', θ_2') belong to $\Omega^{2k-1}(i) = \Omega^{2k-1}(\mathcal{M}) \oplus \Omega^{2k-2}(\mathcal{N})$ rather than $\Omega_{\mathrm{EL}}^{2k-1}(i, \omega)$. The remaining explanations are similar in principle.

III. THE HIGHEST EULER-LAGRANGE COHOMOLOGY AND VOLUME-PRESERVING SYSTEMS

Let us focus on the issues relevant to the volume-preserving systems.

First, it can be easily proved that, for each $k = 1, \ldots, n - 2$ and each point $x \in \mathcal{M}$, a 2-form $\alpha\big|_x \in \Lambda_2(T_x^*\mathcal{M})$ satisfies $\alpha\big|_x \wedge \omega^k = 0$ iff $\alpha\big|_x = 0$. As a consequence, a smooth 2-form, $\alpha \in \Omega^2(\mathcal{M})$ satisfies $\alpha \wedge \omega^k = 0$ iff $\alpha = 0$ everywhere. In other words, the linear

maps

$$\iota_k : \Omega^2(\mathcal{M}) \longrightarrow \Omega^{2k+2}(\mathcal{M})$$

$$\alpha \longmapsto \alpha \wedge \omega^k, \quad 1 \leqslant k \leqslant n - 2 \ .$$

are injective. Thus we can obtain that

$$\mathcal{X}_{\mathrm{S}}^{(1)}(\mathcal{M}, \omega) = \mathcal{X}_{\mathrm{S}}^{(3)}(\mathcal{M}, \omega) = \ldots = \mathcal{X}_{\mathrm{S}}^{(2n-3)}(\mathcal{M}, \omega) \subseteq \mathcal{X}_{\mathrm{S}}^{(2n-1)}(\mathcal{M}, \omega).$$

Especially, when $k = n - 2$, the linear map

$$\iota = \iota_{n-2} : \Omega^2(\mathcal{M}) \longrightarrow \Omega^{2n-2}(\mathcal{M})$$

$$\alpha \longmapsto \alpha \wedge \omega^{n-2} \tag{30}$$

is an isomorphism. If $n = 2$, we use the convention that $\omega^0 = 1$, namely, $\iota = \mathrm{id} : \alpha \longmapsto \alpha$.
Then we can define a linear map ϕ making the following diagram commutative:

$$
\begin{array}{ccc}
\Omega^2(\mathcal{M}) & \xrightarrow{\ \iota\ } & \Omega^{2n-2}(\mathcal{M}) \\
\phi \downarrow & & \mathrm{d} \downarrow \\
\mathcal{X}_{\mathrm{H}}^{(2n-1)}(\mathcal{M}, \omega) & \xrightarrow{\ \nu_n\ } & B^{2n-1}(\mathcal{M}),
\end{array}
\tag{31}
$$

where ν_n the linear isomorphism sending a vector field $\boldsymbol{X}$ to a $(2n-1)$-form $-i_{\boldsymbol{X}}(\omega^n)$ [8?
].

It should be mentioned that, equivalently, given a 2-form

$$\alpha = \frac{1}{2} Q_{ij} \, \mathrm{d}q^i \wedge \mathrm{d}q^j + A_j^i \, \mathrm{d}p_i \wedge \mathrm{d}q^j + \frac{1}{2} P^{ij} \, \mathrm{d}p_i \wedge \mathrm{d}p_j \tag{32}$$

where Q_{ij} and P^{ij} satisfy

$$Q_{ji} = -Q_{ij}, \qquad P^{ji} = -P^{ij}, \tag{33}$$

the vector field

$$\phi(\alpha) = (\nu_n^{-1} \circ \mathrm{d} \circ \iota)(\alpha) = \nu_n^{-1}(\mathrm{d}\alpha \wedge \omega^{n-2})$$

is in $\mathcal{X}_{\mathrm{H}}^{(2n-1)}(\mathcal{M}, \omega)$.

For convenience, we set

$$\boldsymbol{X} = n(n-1) \, \phi(\alpha) = n(n-1) \, \nu_n^{-1}(\mathrm{d}\alpha \wedge \omega^{n-2}),$$

namely, $i_{\boldsymbol{X}}(\omega^n) = -n(n-1)\,(\mathrm{d}\alpha)\wedge\omega^{n-2}$. It is easy to obtain that

$$\boldsymbol{X} = \left(\frac{\partial P^{ij}}{\partial q^j} + \frac{\partial A^j_j}{\partial p_i} - \frac{\partial A^i_j}{\partial p_j}\right)\frac{\partial}{\partial q^i} + \left(\frac{\partial Q_{ij}}{\partial p_j} - \frac{\partial A^j_j}{\partial q^i} + \frac{\partial A^j_i}{\partial q^j}\right)\frac{\partial}{\partial p_i}. \tag{34}$$

In fact, it can be easily obtained that

$$\begin{aligned}
\mathrm{d}\iota(\alpha) &= \mathrm{d}\alpha\wedge\omega^{n-2}\\
&= \left(\frac{1}{2}\frac{\partial Q_{jk}}{\partial p_i} + \frac{\partial A^i_j}{\partial q^k}\right)\mathrm{d}p_i\wedge\mathrm{d}q^j\wedge\mathrm{d}q^k\wedge\omega^{n-2}\\
&\quad + \left(\frac{1}{2}\frac{\partial P^{ij}}{\partial q^k} - \frac{\partial A^i_k}{\partial p_j}\right)\mathrm{d}p_i\wedge\mathrm{d}p_j\wedge\mathrm{d}q^k\wedge\omega^{n-2}.
\end{aligned}$$

By virtue of the following two equations

$$\begin{aligned}
\mathrm{d}p_i\wedge\mathrm{d}p_j\wedge\mathrm{d}q^k\wedge\omega^{n-2} &= \frac{\delta^k_j}{n-1}\,\mathrm{d}p_i\wedge\omega^{n-1} - \frac{\delta^k_i}{n-1}\,\mathrm{d}p_j\wedge\omega^{n-1},\\
\mathrm{d}p_i\wedge\mathrm{d}q^j\wedge\mathrm{d}q^k\wedge\omega^{n-2} &= \frac{\delta^j_i}{n-1}\,\mathrm{d}q^k\wedge\omega^{n-1} - \frac{\delta^k_i}{n-1}\,\mathrm{d}q^j\wedge\omega^{n-1},
\end{aligned}$$

we can write $\mathrm{d}\iota(\alpha)$ as the inner product of certain a vector with ω^n. Comparing it with

$$\mathrm{d}\iota(\alpha) = \frac{1}{n(n-1)}\,\nu_n(\boldsymbol{X}) = -\frac{1}{n(n-1)}\,i_{\boldsymbol{X}}(\omega^n), \tag{35}$$

we can obtain the expression of $\boldsymbol{X}$, as shown in eq. (34).

Since both ι and ν_n are linear isomorphisms, we can see from the commutative diagram (31) that, for each 2-form α on $\mathcal{M}$ as in eq. (32), the vector field $\boldsymbol{X}$ in eq. (34) belongs to $\mathcal{X}^{(2n-1)}_{\mathrm{H}}(\mathcal{M},\omega)$; Also, for each $\boldsymbol{X}\in\mathcal{X}^{(2n-1)}_{\mathrm{H}}(\mathcal{M},\omega)$ there exists the 2-form α on $\mathcal{M}$ satisfying eq. (34). But there may be several 2-forms that are mapped to the same vector field $\boldsymbol{X}$. For example, the vector field $\boldsymbol{X}$ in (34) is invariant under the transformation

$$\alpha\longmapsto\alpha+\theta \tag{36}$$

where θ is a closed 2-form.

Note that for the vector field $\boldsymbol{X}\in\mathcal{X}^{(2n-1)}_{\mathrm{H}}(\mathcal{M},\omega)$, the 2-form α is a globally defined. If $\boldsymbol{X}\in\mathcal{X}^{(2n-1)}_{\mathrm{S}}(\mathcal{M},\omega)$, such a 2-form cannot globally be given if $H^{(2n-1)}_{\mathrm{dR}}(\mathcal{M})$ is nontrivial. In this case, α can be still found as a locally defined 2-form. Then the relation between $\boldsymbol{X}$ and the locally defined 2-form α, eq. (34), is valid only on an open subset of $\mathcal{M}$.

It is clear that no matter whether $\boldsymbol{X}$ belongs to $\mathcal{X}_\mathrm{H}^{(2n-1)}(\mathcal{M}, \omega)$ or $\mathcal{X}_\mathrm{S}^{(2n-1)}(\mathcal{M}, \omega)$, i.e., whether the 2-form α is globally or locally defined, the flow of $\boldsymbol{X}$ can be always obtained provided that the general solution of the following equations can be obtained:

$$\begin{aligned}
\dot{q}^{\,i} &= \frac{\partial P^{ij}}{\partial q^j} + \frac{\partial A_j^j}{\partial p_i} - \frac{\partial A_j^i}{\partial p_j}, \\
\dot{p}_i &= \frac{\partial Q_{ij}}{\partial p_j} - \frac{\partial A_j^j}{\partial q^i} + \frac{\partial A_i^j}{\partial q^j}.
\end{aligned} \tag{37}$$

This is just the general form of the equations of a volume-preserving mechanical system on a symplectic manifold $(\mathcal{M}, \omega)$.

It should be pointed out that as volume-preserving systems the ordinary Hamiltonian systems are included. There should be a 2-form for the Hamiltonian system, in fact, one of such a 2-form can be selected as

$$\alpha = \frac{1}{n-1} H \omega. \tag{38}$$

Substituting this 2-form into eqs. (37), they turn to be the ordinary canonical equations with H as the Hamiltonian.

Let us now consider the non-Hamiltonian linear system without potential:

$$\ddot{q}^{\,i} = -k_{ij}\, q^j \tag{39}$$

on $\mathbb{R}^n$ with constant coefficients k_{ij} if $k_{ij} \neq k_{ji}$.

It is obvious that eqs. (39) can be turned into the form:

$$\dot{q}^{\,i} = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i} - a_{ij}\, q^j \tag{40}$$

with

$$\begin{aligned}
H &= \frac{1}{2} \delta^{ij}\, p_i p_j + \frac{1}{2}\, s_{ij}\, q^i q^j, \\
s_{ij} &= \frac{1}{2}\, (k_{ij} + k_{ji}), \qquad a_{ij} = \frac{1}{2}\, (k_{ij} - k_{ji}).
\end{aligned} \tag{41}$$

When one of a_{ij} is nonzero, the system (40) is just a non-Hamiltonian system with non-potential force on $\mathbb{R}^{2n}$. In fact, the corresponding vector field

$$\boldsymbol{X} = \boldsymbol{X}_H - a_{ij}\, q^j\, \frac{\partial}{\partial p_i}$$

of (40) is not even a symplectic vector field since the Lie derivative with respect to this vector field does not preserve the symplectic form. However, the linear system (40) always preserves the volume form of the phase space because $\mathcal{L}_{\boldsymbol{X}}(\omega^n) = 0$.

If we take

$$\alpha = \frac{1}{n-1} H \, \omega - \frac{1}{2} p_k q^k \, a_{ij} \, dq^i \wedge dq^j, \tag{42}$$

then the general equations (37) turn into eqs. (40). Therefore the 2-form α in eq. (42) is the general 2-form corresponding to a linear system without potential.

It should be mentioned that even a conservative system can be transformed to be a non-Hamiltonian linear system. For example, two 1-dimensional linearly coupled oscillators may construct such a system:

$$m_1 \, \ddot{q}^1 = -k \, (q^2 - q^1), \qquad m_2 \, \ddot{q}^2 = -k \, (q^1 - q^2).$$

Obviously, such a system satisfies Newton's laws. But, it is not a Hamiltonian system if $m_1 \neq m_2$: Let $k_{11} = -k/m_1$, $k_{12} = k/m_1$, $k_{21} = k/m_2$ and $k_{22} = -k/m_2$. Then it is a system as described by eqs. (39). When $m_1 \neq m_2$, we have a system satisfying $k_{12} \neq k_{21}$.

IV.   LIOUVILLE'S THEOREM AND ITS GENERALIZATIONS

A. $H_{EL}^{(1)}(\mathcal{M})$ and Symplectic Liouville's Theorem

It is well known that for a given differentiable vector field X on $\mathcal{M}$, the corresponding flow is defined by the 1-parameter transformation group on $(\mathcal{M}, \omega)$

$$\varphi^t : \mathcal{M} \ \to \ \mathcal{M}, \quad p \mapsto q = \varphi^t(p),$$
$$\varphi^t_* : T\mathcal{M} \ \to \ T\mathcal{M}, \quad X_p \mapsto X_q = \varphi^t_* X_p$$

such that $\varphi_t(p)$ is the integral curve of X from $p \in \mathcal{M}$. The 1-parameter transformation group $\{\varphi^t\} \in Diff(\mathcal{M})$ is called the flow with respect to X.

The *phase flow* is defined by the 1-parameter transformation group on $(\mathcal{M}, \omega)$

$$g^t : (p_i(0), q^j(0)) \mapsto (p_i(t), q^j(t)), \quad \forall t \in [0, 1],$$

where $p_i(t), q^j(t)$ are solutions of canonical equations. Namely, $X \in \mathcal{X}_H(\mathcal{M}, \omega)$, $g^t \in Diff_H(\mathcal{M})$.

The famous **Liouville's theorem** states[1]: The phase flow preserves volume. Namely, for any region $D \subset \mathcal{M}$,

$$\text{volume of } g^t D = \text{volume of } D.$$

In order to distinguish with its generalizations via the Euler-Lagrange cohomology groups, we call it the **Hamiltonian Liouville's theorem**, which is corresponding to the image of the first Euler-Lagrange cohomology group.

However, it is also well known that the symplectic flow generated by a symplectic vector field preserves the volume as well [9] and the symplectic vector field is in the kernel of the first Euler-Lagrange cohomology group $H_{EL}^{(1)}(\mathcal{M}, \omega)$. Eventually, the 1-parameter transformation group on $(\mathcal{M}, \omega)$, $\{f^t\} \in Diff_S(\mathcal{M}, \omega)$, is called a *symplectic flow.* If

$$f_*^t : X \mapsto X_t := f_*^t X, \quad X_{t=0} = X, \ t \in [0, 1] \tag{43}$$

where $X_t \in \mathcal{X}_S^{(1)}(\mathcal{M}, \omega), \forall t \in [0, 1]$, i.e. they are *symplectic vectors.* Thus, we may generalize famous Liouville's theorem to its symplectic counterpart.

Theorem 6 (Symplectic Liouville's Theorem). *The symplectic flow preserves volume. Namely, for any region $D \subset \mathcal{M}$,*

$$\text{volume of } f^t D = \text{volume of } D.$$

This states that the necessary and sufficient condition for the conservation law of volume of $D \subset \Gamma$ under symplectic flow f^t is the Euler-Lagrange 1-form on D is closed.

It is easy to prove this theorem. Since

$$v(t) := \int_{D(t)} \tau, \qquad D(t) = f^t D(0),$$

where $\tau = \frac{1}{n!} \omega^n$ is the volume element. Since

$$\int_{D(t)} \tau = \int_{f^t D(0)} \tau = \int_{D(0)} f^{t*} \tau,$$

and for the symplectic map f^t, by definition we have $f^{t*}\omega = \omega$. This leads to $f^{t*}\tau = \tau$, from which it immediately follows what we want to get: $v(t) = v(0) : \ \int_{D(t)} \tau = \int_{D(0)} \tau$. Thus, the proof is completed.

B. $H_{EL}^{(2k-1)}(\mathcal{M},\omega)$ and Generalized Liouville's Theorem

Let us now consider how to generalize the couple of the symplectic and Hamiltonian Liouville's theorems further.

As was noted, the symplectic and Hamiltonian vector fields generating the symplectic and phase flows correspond to the kernel and image of the first Euler-Lagrange cohomology, respectively. Therefore, it is reasonable to generalize these theorems according to the higher Euler-Lagrange cohomology groups $H_{EL}^{(2k-1)}(\mathcal{M},\omega), 1 < k \leqslant n$. For the kernel and image of each of them, there should be a couple of the $(2k-1)$-degree symplectic(-like) and Hamiltonian(-like) Liouville's theorems, which claim that the $(2k-1)$-degree symplectic(-like) and Hamiltonian(-like) flows, $f_S^{t(2k-1)}$ and $f_H^{t(2k-1)}$ generated by the $(2k-1)$-degree symplectic(-like) and Hamiltonian(-like) vector fields, $\mathcal{X}_S^{(2k-1)}$ and $\mathcal{X}_H^{(2k-1)}$, preserve the volume, respectively. For the highest one, the $(2n-1)$st Euler-Lagrange cohomology group $H_{EL}^{(2n-1)}(\mathcal{M},\omega)$, its kernel and image characterize directly the symplectic(-like) and Hamiltonian(-like) volume-preserving vector fields, respectively. Consequently, the Hamiltonian(-like) volume-preserving Liouville's theorem holds only if the general equations of volume-preserving being satisfied, while the symplectic(-like) volume-preserving Liouville's theorem holds if and only if the flows $f_S^{t(2n-1)}$ are generated by the symplectic(-like) volume-preserving vector fields $\mathcal{X}_S^{(2n-1)}$ and the general equations of volume-preserving being only locally satisfied if this cohomology group is not trivial.

For a given k, $1 \leqslant k < n$, let us first consider a symplectic(-like) vector field $\boldsymbol{X} \in \mathcal{X}_S^{(2k-1)}(\mathcal{M},\omega)$. Assume that its flow is $f_S^{t(2k-1)}$. As stated previously, $\boldsymbol{X}$ is a symplectic vector field, in fact. Therefore we have $\mathcal{L}_{\boldsymbol{X}}\omega = 0$, $\mathcal{L}_{\boldsymbol{X}}\omega^2 = 0$, ..., $\mathcal{L}_{\boldsymbol{X}}\omega^n = 0$. By the definition of the Lie derivatives, we have

$$(f_S^{t(2k-1)})^*\omega = \omega, \quad (f_S^{t(2k-1)})^*\omega^2 = \omega^2, \quad \ldots, (f_S^{t(2k-1)})^*\omega^n = \omega^n \tag{44}$$

for each possible parameter t. For a Hamiltonian(-like) vector field $\boldsymbol{X} \in \mathcal{X}_H^{(2k-1)}(\mathcal{M},\omega)$, it is almost the same except the flow $f_S^{t(2k-1)}$ should be replaced by $f_H^{t(2k-1)}$.

Now consider a $2l$-dimensional orientable submanifold $i : \sigma^{2l} \hookrightarrow \mathcal{M}$ where $1 < 2l \leqslant 2n$. It is possible that $i^*\omega^l = 0$. For example, when $2l \leqslant n$ and σ^{2l} lies in a Lagrangian submanifold, the $2l$-form $i^*\omega^l$ is zero. When it is nonzero, $\frac{1}{l!} i^*\omega^l$ can be treated as a volume form on σ^{2l}. In this case the volume of the submanifold is $\frac{1}{l!} \int_{\sigma^{2l}} \omega^l$. Let $f_L^{t(2k-1)}(\sigma^{2l})$, $L = S, H$, be the

image of σ^{2l} mapped by the corresponding flow $f_L^{t(2k-1)}$, respectively. Then we have

$$\frac{1}{l!} \int_{f_L^{t(2k-1)}(\sigma^{2l})} \omega^l = \frac{1}{l!} \int_{\sigma^{2l}} (f_L^{t(2k-1)})^* \omega^l = \frac{1}{l!} \int_{\sigma^{2l}} \omega^l, \quad L = S, H. \tag{45}$$

That is, the volume of the submanifold σ^{2l} will be preserved when it is sent to another one along the flow $f_{S,H}^{t(2k-1)}$ of a symplectic or Hamiltonian(-like) vector filed, respectively.

However, for a symplectic(-like), including the Hamiltonian(-like), volume-preserving vector field $\boldsymbol{X} \in \mathcal{X}_S^{(2n-1)}(\mathcal{M}, \omega), \mathcal{X}_H^{(2n-1)}(\mathcal{M}, \omega)$, the forms $\omega, \omega^2, \ldots, \omega^{n-1}$ are not necessarily preserved. In this case we can consider only a region $D \subset \mathcal{M}$. It can be similarly discussed that

$$\frac{1}{n!} \int_{f_L^{t(2n-1)}(D)} \omega^n = \frac{1}{n!} \int_D \omega^n, \quad L = S, H. \tag{46}$$

In conclusion, the sequence of area preserving laws (1) in classical mechanics should also be generalized to a couple of sequences of the area preserving laws (45), (46) according to the Euler-Lagrange cohomological groups. Thus, we have

Theorem 7 (Generalized Liouville's Theorem). *Given a vector field $\boldsymbol{X}_L \in \mathcal{X}_S^{(2k-1)}(\mathcal{M}, \omega)$, $(L = S, H)$ on $(\mathcal{M}, \omega)$, its flow $f_L^{t(2k-1)}$, $(L = S, H)$, preserves the volume. Namely, for any region $D \subset \mathcal{M}$,*

$$\textit{volume of } f_L^{t(2k-1)} D = \textit{volume of } D, \quad L = S, H.$$

As was mentioned, for the Hamiltonian(-like) sequence $(L = H)$ of the conservation laws, the corresponding equations of motion hold. Namely, this sequence of conservation laws are "on shell". For the symplectic(-like) sequence $(L = S)$ of these conservation laws with $k < n$, the relevant Hamiltonian in the equations of motion are defined locally if the Euler-Lagrange cohomology group $H_{\text{EL}}^{(1)}(\mathcal{M}, \omega)$ is not trivial. For the symplectic(-like) volume systems $(L = S$ and $k = n)$, there will be no Hamiltonians (even defined locally). Instead, they are described by some 2-forms defined patch by patch provided that $H_{\text{EL}}^{(2n-1)}(\mathcal{M}, \omega)$ is nontrivial.

It should be mentioned that this also generalizes the famous Noether's theorem in classical mechanics. For a given Hamiltonian system with symmetries, Noether's theorem always relates all conservation laws with equations of motion of the system since the action is invariant under the symmetries. For the symplictic(-like) Liouville's theorems, however, they

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relate the corresponding simplicity conservation laws directly with the kernel of the relevant Euler-Lagrange cohomology rather than equations of motion of the original mechanical systems.

V.   CONCLUDING REMARKS

In this paper, we have introduced the definition of the Euler-Lagrange cohomology groups $H_{\mathrm{EL}}^{(2k-1)}(\mathcal{M}, \omega)$, $1 \leqslant k \leqslant n$, on symplectic manifolds $(\mathcal{M}^{2n}, \omega)$ and studied their relations with other cohomologies as well as some of their properties [6], [7], [8]. It is shown that, for $k = 1, n$, $H_{\mathrm{EL}}^{(1)}(\mathcal{M}, \omega)$ and $H_{\mathrm{EL}}^{(2n-1)}(\mathcal{M}, \omega)$ are isomorphic to the de Rham cohomology $H_{\mathrm{dR}}^{(1)}(\mathcal{M})$ and $H_{\mathrm{dR}}^{(2n-1)}(\mathcal{M})$, respectively. On the other hand, generally $H_{\mathrm{EL}}^{(2k-1)}(\mathcal{M}, \omega)$, $1 < k < n$, is isomorphic neither to the de Rham cohomology $H_{\mathrm{dR}}^{(2k-1)}(\mathcal{M})$ nor to the harmonic cohomology on $(\mathcal{M}^{2n}, \omega)$. And they are also different from each other in general. To our knowledge, these Euler-Lagrange cohomology groups on $(\mathcal{M}^{2n}, \omega)$ have not yet been introduced systematically before. It is significant to know whether there are some more important roles played by these cohomology groups to the symplectic manifolds.

The ordinary canonical equations in Hamilton mechanics correspond to 1-forms that represent trivial element in the first Euler-Lagrange cohomology $H_{\mathrm{EL}}^{(1)}(\mathcal{M}, \omega)$ on the phase space. Analog to this property, the general volume-preserving Hamiltonian equations on phase space are presented, from the cohomological point of view, in terms of forms which represent the trivial element in the highest Euler-Lagrange cohomology group $H_{\mathrm{EL}}^{(2n-1)}(\mathcal{M}, \omega)$. And the ordinary canonical equations in Hamilton mechanics become their special cases. It is of course interesting to see what about the representatives in the trivial elements of other Euler-Lagrange cohomology groups and whether they also lead to some meaningful dynamical equations. These problems are under investigation.

We have also introduced the conception of relative Euler-Lagrange cohomology. It is of course interesting to see its applications in mechanics and physics.

The first Euler-Lagrange cohomology group has been introduced in order to consider its time-discrete version in the study on the discrete mechanics including the symplectic algorithm [3, 4]. Although the first Euler-Lagrange cohomology is isomorphic to the first de Rham cohomology, its time discrete version is still intriguing and plays an important role in the symplectic algorithm. In addition, it has also been introduced in the field theory and

their discrete versions of independent variables. The latter is closely related to the multi-symplectic algorithm [3, 4]. It is of course significant to introduce the higher Euler-Lagrange cohomology groups in these fields and to explore their applications. Furthermore, since the general volume-preserving equations have been introduced on symplectic manifold from the cohomological point of view, it is meaningful to investigate their time-discrete version and study its relation with the volume-preserving algorithm.

Based on the Euler-Lagrange cohomology groups and Hamiltonian(-like) volume-preserving equations on symplectic manifold $(\mathcal{M}, \omega)$, we have generalized famous Liouville's theorem in classical mechanics. It has been first generalized to the symplectic flows generated by the symplectic vector fields, the kernel of the first Euler-Lagrange cohomology group, i.e. to *symplectic Liouville's theorem*. Then, we have further generalized the symplectic and Hamiltonian Liouville's theorems to the higher ones corresponding to the kernel and image of each higher Euler-Lagrange cohomology group $H_{\mathrm{EL}}^{(2k-1)}(\mathcal{M})$, $1 < k \leqslant n$ on $(\mathcal{M}, \omega)$, respectively, i.e. the symplectic(-like) and Hamiltonian(-like) Liouville's theorems.

In all these Liouville's theorems, the symplectic(-like) (for $1 \leqslant k \leqslant n$) and the Hamiltonian(-like) (for $1 < k \leqslant n$) ones have offered a kind of generalizations of the famous Noether's theorem, which does not require the systems to possess actions necessarily in ordinary sense. In other words, the conservation laws may hold with respect to the relevant cohomology and for the classical systems with non-potential forces.

It should be mentioned that this kind of generalization of Noether's theorem may also work for other cohomology groups of the symmetries of the systems on the relevant manifolds. We will explore these aspects in detail elsewhere.

Finally, it should also be pointed out that these Liouville's theorems should play important roles not only in classical and continuous mechanics, field theories but also especially in statistical physics. In fact, the statistical Liouville's theorem, based on Liouville's theorem in mechanics, is one of the fundamental principles in statistical physics. According to our results, however, the generalized Liouville's theorems should offer a way to generalize Liouville's theorem in statistical physics. We should also leave this issue for the forthcoming publication.

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[1] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New-York, 1989.

[2] R. Abraham and J.E. Marsden, *Foundation of Mechanics*, (Second Ed.) Addison-Wesley, 1978.

[3] Han-Ying Guo, Yu-Qi Li and Ke Wu, "On Symplectic and Multisymplectic Structures and Their Discrete Version in Lagrangian Formalism", *Comm. Theor. Phys.* (Beijing) **35** (2001) 703-710.

[4] Han-Ying Guo, Yu-Qi Li, Ke Wu and Shi-Kun Wang, "Difference Discrete Variational Principle, Euler-Lagrange Cohomology and Symplectic, Multisymplectic structures I: Difference Discrete Variational Principle", *Comm. Theor. Phys.* (Beijing), **37** (2002) 1-10; "II: Euler-Lagrange Cohomology", ibid, 129-138; "III: Application to Symplectic and Multisymplectic Algorithms", ibid, 257-264. See, also, the epreprint ArXive: hep-th/0106001.

[5] Han-Ying Guo and Ke Wu, "On variations in discrete mechanics and field theory", *J. Math. Phys.* **44** (2003) 5978-6004. See, also, hep-th/0112239. And references therein.

[6] Bin Zhou, Han-Ying Guo, Jianzhong Pan and Ke Wu, The Euler-Lagrange Cohomology on Symplectic Manifolds and General Volume-Preserving Sustems, *Mod. Phys. Lett.* **A18** (2003) 1911-1924. preprint arXiv: hep-th/0304075.

[7] B. Zhou, H. Y. Guo and K. Wu, *Lett. Math. Phys.* **64** (2003) 235.

[8] Han-Ying Guo, Jianzhong Pan, Ke Wu and Bin Zhou, The Euler-Lagrange Cohomology on Symplectic Manifolds, preprint arXiv: physics/0304074.

[9] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Clar. Press. Oxford, 1998.

[10] A.I. Malcev: On a class of homogeneous spaces, *Izv AN SSSR Ser. Matem.* **3** (1949), 9-32.

[11] M. Raghunathan: *Discrete subgroups of Lie groups*, Springer, Berlin, 1972.

[12] A. Tralle, J. Oprea: *Symplectic Manifolds with no Kähler Structure*, Lecture Notes in Math. **1661**, Springer, Berlin 1997.

[13] D. Yan, Hodge structure on symplectic manifolds, *Adv. in Math.* **120** (1996), 143-154.

[14] J.L. Brylinski: A differential complex for Poisson manifolds, *J. Diff. Geom.* **28** (1988), 93-114.

[15] P. Libermann, C. Marle: *Symplectic Geometry and Analytical Mechanics*, Kluwer, Dordrecht, 1987.

[16] J.L. Koszul: Crochet de Schouten-Nijenhuis et cohomologie, The mathematical heritage of Élie Cartan (Lyon, 1984). *Astrisque* 1985, Numero Hors Serie, 257–271.